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The B-bar method and the limitation principles

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Abstract

The generalized elastic material provides a reference model to cast in a unitary framework many structural models which are based on nonlinear monotone multivalued relations such as viscoelasticity, plasticity and unilateral models. The modified forms of the Hu–Washizu and Hellinger–Reissner principles and the displacement-based variational formulation are recovered for the generalized elastic material starting from a functional in the complete set of state variables. The related limitation principles are derived and their specialization to elasticity and elastoplasticity with mixed hardening are provided. It is shown that the interpolating fields for the pressure and the volumetric strain usually adopted in the B-bar method lead to a limitation principle. Accordingly the same elastic and elastoplastic solutions can be obtained by means of an approximate mixed displacement/pressure variational principle. A second application is concerned with the conditions ensuring the coincidence of the solutions between an approximate two-field mixed formulation and the displacement-based method. Numerical examples are provided to show the coincidence of the solutions obtained from different mixed finite element formulations, in elasticity or elastoplasticity, under the validity of the limitation principles.

1. Introduction

Mixed variational formulations are adopted to develop mixed finite element methods for the treatment of incompressible or nearly incompressible problems. In the elastic range the volume constraint arises from the assumption of an incompressible or almost incompressible behaviour which represents a proper schematization for rubber-like materials, solid propellants, polymers, etc. In elastoplasticity, incompressibility comes from the assumption of an isochoric plastic flow.

For almost incompressible materials, the displacement-based finite element method provides incorrect stress fields, particularly for the pressure, and locks in the sense that there is a loss of accuracy in the computed response as incompressibility is enforced, see e.g. Sussman and Bathe

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(1987). If incompressibility is enforced, it is unreasonable to expect a correct prediction of the stress field since the pressure field is uncoupled from the displacement field and must be calculated from the equilibrium equation. Moreover it is impossible to compute the failure load if the material has an elastic–perfectly plastic behaviour (Nagtegaal et al., 1974).

Multifield variational principles eliminate the difficulties encountered in the single-field variational principle. A modified version of the Hellinger–Reissner principle can be used to provide a displacement/pressure mixed finite element approximation for incompressible linear and nonlinear elastic materials and for elastoplastic problems, see e.g., Key (1969), Nagtegaal et al. (1974), Sussman and Bathe (1987). An alternative approach can be recovered starting from a modified form of the Hu–Washizu principle in which the displacement, the pressure and the volumetric strain are the independent fields. It is named the B-bar method and is often referred to in the computational literature (Simo et al., 1985; Zienkiewicz and Taylor, 1991; Comi and Perego, 1995; Weiss et al., 1996).

Although solutions derived from the variational principle pertaining to the continuum problem coincide, those derived from approximate variational principles are different unless a limitation principle occurs (Stolarski and Belytschko, 1987; Zienkiewicz and Taylor, 1991; Alfano and Marotti de Sciarra, 1996). In this case no additional accuracy has to be expected from a mixed formulation.

The objective of this work is to determine the conditions which lead to the coincidence of the solutions derived from approximate variational formulations depending on different sets of interpolating variables.

To analyse this problem for different structural models from a unitary point of view, a general structural theory can be developed, providing general methods of investigation.

Many constitutive models (such as viscoelasticity, elastoplasticity and unilateral models) can be described in terms of nonlinear monotone multivalued relations. These models can be cast in the unitary framework provided by the generalized elastic material (Romano, 1994) which is based on an extension of the classical potential theory to the case of monotone multivalued operators (Romano et al., 1993b).

The variational formulation in the whole set of state variables is recovered by a direct integration of the structural operator associated with the generalized elastic material. The complete set of limitation principles is provided in this paper with reference to the approximate variational formulations thus generalizing the one stated in Alfano and Marotti de Sciarra (1996).

The limitation principles associated with the Hu–Washizu and Hellinger–Reissner principles and with the displacement-based variational formulation are discussed in detail for the generalized elastic material. They are specialized to approximate variational formulations widely adopted in elasticity and in elastoplasticity without repeating ad hoc reasoning.

In particular we show that the interpolating fields usually adopted in the B-bar method, see e.g., Zienkiewicz and Taylor (1991), Weiss et al. (1996) fulfil a limitation principle so that the displacement/pressure formulation provides the same approximated displacement and pressure fields of the B-bar method.

Moreover a limitation principle provides a new analytical condition to check whether the solutions derived from the approximate mixed displacement/pressure formulation and from the displacement-based method coincide in terms of the displacement field.

The generalized elastic material is then specialized to an elastoplastic behaviour with hardening

(Halphen and Nguyen, 1975; Lubliner, 1990) and the elastoplastic counterparts of the B-bar method and of the mixed displacement/pressure formulation are recovered. The related limitation principle is thus obtained from the general treatment and it is exploited to prove that the interpolating fields assumed in Simo et al. (1985) meet a limitation principle.

For elastic and elastoplastic models numerical examples with different values of the Poisson's ratio are examined. The same solutions are recovered from different mixed finite element formulations since interpolations of state variables which fulfil the limitation principles are considered.

2. Generalized elastic material

Generalized elasticity provides a reference constitutive model which can be referred to in the analysis of a wide class of structural problems involving linear and nonlinear constitutive relations.

Let $\mathscr{D}$ and $\mathscr{S}$ be the dual vector spaces of strain fields ε and of stress fields σ. The Generalized Elastic Material (GEM) has no memory since the relation between dual quantities ε and σ depends on their actual value.

The constitutive relation $\mathscr{E}: \mathscr{D} \to \mathscr{S}$ between strain and stress fields is characterized by the following properties of its graph $\mathscr{G}(\mathscr{E})$:

(i) $\mathscr{G}(\mathscr{E})$ is maximal monotone and conservative,

(ii) $\operatorname{dom}\mathscr{E}$ and $\operatorname{dom}\mathscr{E}^{-1}$ are convex sets.

A pair $(\varepsilon, \sigma) \in \mathscr{D} \times \mathscr{S}$ fulfils the constitutive relation $\mathscr{E}$ if it belongs to the graph $\mathscr{G}(\mathscr{E})$:

$$(\varepsilon, \sigma) \in \mathscr{G}(\mathscr{E}) \Leftrightarrow \sigma \in \mathscr{E}(\varepsilon) \Leftrightarrow \varepsilon \in \mathscr{E}^{-1}(\sigma). \tag{1}$$

Monotonicity of $\mathscr{G}$ implies that the material tangent stiffness must be nonnegative. Maximal monotonicity ensures that the point values of the generalized elastic relation $\mathscr{E}$ and of its inverse $\mathscr{E}^{-1}$ are convex sets.

The multivalued elastic map $\mathscr{E}$ is assumed to be conservative. Accordingly the work associated with any stress field $\sigma \in \mathscr{E}(\varepsilon)$ along any closed polyline Π_ε contained in the domain of $\mathscr{E}$ vanishes:

$$\oint_{\Pi_\varepsilon} \langle \mathscr{E}(\varepsilon), \mathrm{d}\varepsilon \rangle = 0 \quad \forall \Pi_\varepsilon \subseteq \operatorname{dom}\mathscr{E}. \tag{2}$$

The conservativity of $\mathscr{E}$ implies that the multivalued inverse map $\mathscr{E}^{-1}$ is conservative as well. A detailed presentation of the potential theory for monotone multivalued operators can be found in Romano et al. (1993b).

The conservativity of the two maps $\mathscr{E}$ and $\mathscr{E}^{-1}$ ensures the existence of two complementary convex potentials $\Phi: \mathscr{D} \to \Re \bigcup \{+\infty\}$ and $\Psi: \mathscr{S} \to \Re \bigcup \{+\infty\}$ defined on $\operatorname{dom}\mathscr{E}$ and $\operatorname{dom}\mathscr{E}^{-1}$ which are given by:

$$\Phi(\varepsilon) - \Phi(\varepsilon_0) = \int_{\varepsilon_0}^{\varepsilon} \langle \mathscr{E}(\bar{\varepsilon}), \mathrm{d}\bar{\varepsilon} \rangle \qquad \Psi(\sigma) - \Psi(\sigma_0) = \int_{\sigma_0}^{\sigma} \langle \mathscr{E}^{-1}(\bar{\sigma}), \mathrm{d}\bar{\sigma} \rangle,$$

and are set $+\infty$ outside their domains.

In the sequel, we will consider a GEM with a regular constitutive relation, i.e.:

$$\Psi^*(\varepsilon) = \Phi(\varepsilon) = \sup_{\bar{\sigma}\in\mathscr{S}} \{\langle\bar{\sigma},\varepsilon\rangle - \Psi(\bar{\sigma})\} \quad \forall\varepsilon\in\mathscr{D}$$

$$\Phi^*(\sigma) = \Psi(\sigma) = \sup_{\bar{\varepsilon}\in\mathscr{D}} \{\langle\sigma,\bar{\varepsilon}\rangle - \Phi(\bar{\varepsilon})\} \quad \forall\sigma\in\mathscr{S}.$$

The convex functionals Φ^* and Ψ^* are known in convex analysis as the Fenchel's conjugates (Rockafellar, 1970) of Φ and Ψ.

The generalized elastic relation (1) can then be rewritten in terms of the subdifferentials of Φ and Φ^* as:

$$(\varepsilon,\sigma)\in\mathscr{G}(\mathscr{E}) \Leftrightarrow \sigma\in\partial\Phi(\varepsilon) \Leftrightarrow \varepsilon\in\partial\Phi^*(\sigma). \tag{3}$$

From a mechanical point of view the convex potentials $\Phi(\varepsilon)$ and $\Phi^*(\sigma)$ represent the elastic energy and the complementary elastic energy for the GEM.

3. Structural model

Let $\mathscr{U}$ be the linear space of displacement fields and $\mathscr{F}$ be the dual linear space of external forces which are given by the sum of the applied load l and of the reactions r of external constraints.

In a geometrically linear range, the equilibrium equation and the compatibility relation are expressed as follows:

$$l+r = T'\sigma \qquad \varepsilon = Tu$$

where the linear equilibrium operator $T'\colon \mathscr{S}\mapsto\mathscr{F}$ is dual of the linear kinematic operator $T\colon \mathscr{U}\mapsto\mathscr{D}$ (Panagiotopoulos, 1985).

Introducing the concave potential $\Upsilon\colon \mathscr{U}\mapsto\Re\bigcup\{-\infty\}$, the external constitutive relation between reactions and displacement fields can be written in the general form:

$$r\in\partial\Upsilon(u) \Leftrightarrow u\in\partial\Upsilon^*(r)$$

where Υ^* is the conjugate of Υ.

In the case of external frictionless bilateral constraints with an imposed displacement w, admissible displacement fields u belong to the affine variety $L = w+L_0 \subseteq \mathscr{U}$, being L_0 the subspace of the admissible displacement fields which satisfy the boundary conditions.

The subspace of external reactions is $\Re = L_0^\perp \subseteq \mathscr{F}$ where $L_0^\perp$ represents the orthogonal complement of the subspace L_0. The expressions of the concave potentials Υ and Υ^* are then given by:

$$\Upsilon(u) = |\overline{}|_{L_0}(u-w) \qquad \Upsilon^*(r) = |\overline{}|_{L_0^\perp}(r) + \langle r,w\rangle$$

where $|\overline{}|$ is the concave indicator.

The structural problem for the GEM is thus governed by the following set of relations:

$$\begin{aligned} T'\sigma &= l+r && \text{static equilibrium}\\ Tu &= \varepsilon && \text{kinematic compatibility}\\ \sigma &\in \partial\Phi(\varepsilon) && \text{generalized constitutive relation}\\ u &= \partial\Upsilon^*(r) && \text{external constraint.} \end{aligned} \tag{4}$$

3.1. Variational formulations

We can arrange relations (4) to build up a global multivalued structural operator governing the generalized model whose explicit form is:

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & T' & \mathbf{0} & -\mathbf{I}_{\mathscr{F}} \\ T & \mathbf{0} & -\mathbf{I}_{\mathscr{D}} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{\mathscr{S}} & \partial\Phi & \mathbf{0} \\ -\mathbf{I}_{\mathscr{U}} & \mathbf{0} & \mathbf{0} & \partial\Upsilon^* \end{bmatrix}.$$

The structural operator **A** is given by the sum of a linear symmetric operator, and hence conservative, and of two conservative monotone multivalued operators: $\partial\Phi$ nondecreasing in ε and $\partial\Upsilon^*$ nonincreasing with respect to r. The relevant potential Ω can be directly obtained by integrating **A** along a ray individuated by the point $(u, \sigma, \varepsilon, r)$, see Romano et al. (1993a) for finite-step elastoplasticity and Romano et al. (1992) for nonlinear models. The expression of Ω is given by:

$$\Omega(u, \sigma, \varepsilon, r) = \Phi(\varepsilon) + \Upsilon^*(r) + \langle \sigma, Tu \rangle - \langle l + r, u \rangle - \langle \sigma, \varepsilon \rangle \tag{5}$$

and turns out to be linear in (u, σ), convex in ε and concave in r for any applied load l.

Hence we have:

Proposition 3.1. A vector $(u, \sigma, \varepsilon, r)$ is a saddle point solution of the problem

$$\operatorname*{stat}_{u,\sigma} \min_{\varepsilon} \max_{r} \Omega(u, \sigma, \varepsilon, r)$$

if and only if it is a solution of the generalized structural problem. □

Remark 3.1. It is worth noting that the functional Ω is obtained by a direct integration of the structural operator **A** associated with (4) and thus its stationarity is equivalent to the generalized model (4). ■

A family of variational principles, with different numbers of unknowns, can be derived from the potential Ω by enforcing the fulfilment of field equations and of constitutive relations. All these principles provide the same solution of the structural problem (4) if no approximation of the ambient spaces is introduced.

3.1.1. A three-field variational principle

To obtain a variational formulation in terms of the spherical part of the stress and of the strain fields, the variational principle depending on the three independent fields (u, σ, ε) is first derived.

The external constraint relation $(4)_4$ can be equivalently expressed in terms of the following Fenchel's equality:

$$\Upsilon^*(r) + \Upsilon(u) = \langle r, u \rangle$$

which, substituted in the expression of the potential Ω, yields:

Proposition 3.2. Generalized Hu–Washizu principle. A vector (u, σ, ε) is a solution of the minimization problem:

$$\operatorname*{stat}_{\sigma} \min_{u,\varepsilon} \Omega_1(u, \sigma, \varepsilon)$$

where:

$$\Omega_1(u, \sigma, \varepsilon) = \Phi(\varepsilon) - \Upsilon(u) + \langle \sigma, Tu \rangle - \langle l, u \rangle - \langle \sigma, \varepsilon \rangle$$

if and only if it is a solution of the generalized structural problem. □

3.2. Spherical and deviatorical split

The volumetric and deviatoric responses are in general coupled. However in many situations we may assume that the generalized energy Φ can be additively decomposed into two parts depending separately on the volumetric and deviatoric strain fields:

$$\Phi(\varepsilon) = \Phi_1(P_D\varepsilon) + \Phi_2(P_S\varepsilon), \tag{6}$$

see Simo et al. (1985) in finite deformations and Weiss et al. (1996) with respect to highly deformable biological soft tissue. The operators P_S and P_D are the projectors onto the spaces of spherical and deviatoric strain fields, respectively.

The conjugate Φ^* of the generalized energy (6) is given by the sum of the conjugate functionals of Φ_1 and Φ_2. In fact, we have:

$$\begin{aligned}\Phi^*(\sigma) &= \sup_{\bar{\varepsilon}\in\mathscr{D}} \{\langle \sigma, \bar{\varepsilon} \rangle - \Phi(\bar{\varepsilon})\} \\ &= \sup_{P_D\bar{\varepsilon}, P_S\bar{\varepsilon}} \{\langle P_D\sigma, P_D\bar{\varepsilon} \rangle + \langle P_S\sigma, P_S\bar{\varepsilon} \rangle - \Phi_1(P_D\bar{\varepsilon}) - \Phi_2(P_S\bar{\varepsilon})\} \\ &= \Phi_1^*(P_D\sigma) + \Phi_2^*(P_S\sigma), \end{aligned} \tag{7}$$

where the orthogonality condition between deviatoric and spherical fields has been exerted. An example of the determination of Φ_2^* in elasticity is reported in Reissner (1984).

Variational formulations for the GEM in terms of the spherical part of the stress and of the strain fields can be recovered from the generalized Hu–Washizu principle.

In fact by enforcing in the expression of the functional Ω_1 the deviatoric part of the kinematic compatibility relation $(4)_2$:

$$P_D Tu = P_D\varepsilon$$

and by replacing the generalized energy Φ by the additive decomposition (6) we get:

Proposition 3.3. Modified Hu–Washizu principle. A vector $(u, P_S\sigma, P_S\varepsilon)$ is a solution of the minimization problem:

$$\operatorname*{stat}_{P_S\sigma} \min_{u,P_S\varepsilon} \Omega_2(u, P_S\sigma, P_S\varepsilon)$$

where:

$$\Omega_2(u, P_S\sigma, P_S\varepsilon) = \Phi_1(P_D Tu) + \Phi_2(P_S\varepsilon) - \Upsilon(u) + \langle P_S\sigma, P_S(Tu - \varepsilon) \rangle - \langle l, u \rangle$$

if and only if it is a solution of the generalized structural problem. □

The structural model derived from the above variational principle is obtained by enforcing that $(u, P_S\sigma, P_S\varepsilon)$ is a stationarity point for Ω_2, i.e.:

$$(0,0,0) \in \partial\Omega_2(u, P_S\sigma, P_S\varepsilon)$$

which is, by definition:

$$\begin{aligned} &0 \leqslant \mathrm{d}\Omega_2(u, P_S\sigma, P_S\varepsilon; \bar{u}) && \forall \bar{u} \in \mathscr{U} \\ &0 = \mathrm{d}\Omega_2(u, P_S\sigma, P_S\varepsilon; P_S\bar{\sigma}) && \forall P_S\bar{\sigma} \in \mathscr{S} \\ &0 \leqslant \mathrm{d}\Omega_2(u, P_S\sigma, P_S\varepsilon; P_S\bar{\varepsilon}) && \forall P_S\bar{\varepsilon} \in \mathscr{D}. \end{aligned}$$

The evaluation of the above directional derivatives yields the following expressions:

$$\begin{aligned} &\mathrm{d}\Phi_1(P_D Tu; P_D T\bar{u}) + \langle P_S\sigma, P_S T\bar{u}\rangle \geqslant \langle l, \bar{u}\rangle \quad u - w \in L_0 && \forall \bar{u} \in L_0 \\ &\langle P_S\bar{\sigma}, P_S(Tu - \varepsilon)\rangle = 0 && \forall P_S\bar{\sigma} \in \mathscr{S} \\ &\mathrm{d}\Phi_2(P_S\varepsilon; P_S\bar{\varepsilon}) - \langle P_S\sigma, P_S\bar{\varepsilon}\rangle \geqslant 0 && \forall P_S\bar{\varepsilon} \in \mathscr{D} \end{aligned} \tag{8}$$

where the condition of external bilateral constraints has been enforced. They provide an alternative weak formulation of the structural model (4). In fact relation $(8)_1$ is the weak form of the equilibrium equation in which the deviatoric stress field is obtained from the generalized energy Φ_1 in terms of a deviatoric compatible strain field. Equation $(8)_2$ provides the weak form of the spherical part of the compatibility condition and inequality $(8)_3$ represents the weak form of the spherical part of the constitutive relation.

A two field variational principle can now be derived from Ω_2. If the spherical part of the generalized constitutive relation $(4)_3$ is fulfilled then, in terms of Fenchel's equality, it turns out to be:

$$\Phi_2(P_S\varepsilon) + \Phi_2^*(P_S\sigma) = \langle P_S\sigma, P_S\varepsilon\rangle.$$

This equality can be inserted in the expression of Ω_2 to get:

Proposition 3.4. Modified Hellinger–Reissner principle. A vector $(u, P_S\sigma)$ is a saddle point solution of the problem

$$\min_u \max_{P_S\sigma} \Omega_3(u, P_S\sigma)$$

where:

$$\Omega_3(u, P_S\sigma) = \Phi_1(P_D Tu) - \Phi_2^*(P_S\sigma) - \Upsilon(u) + \langle P_S\sigma, P_S Tu\rangle - \langle l, u\rangle$$

if and only if it is a solution of the generalized structural problem. □

The structural model associated with the modified form of the Hellinger–Reissner principle is obtained by enforcing the stationarity of the potential Ω_3:

$$(0,0) \in \partial\Omega_3(u, P_S\sigma)$$

which is equivalent, by definition, to the inequalities:

$$0 \leqslant \mathrm{d}\Omega_3(u, P_S\sigma; \bar{u}) \qquad \forall \bar{u} \in \mathscr{U}$$
$$0 \geqslant \mathrm{d}\Omega_3(u, P_S\sigma; P_S\bar{\sigma}) \quad \forall P_S\bar{\sigma} \in \mathscr{S}$$

The evaluation of the above directional derivatives yields the following expressions:

$$\begin{aligned} &\mathrm{d}\Phi_1(P_D Tu; P_D T\bar{u}) + \langle P_S\sigma, P_S T\bar{u}\rangle \geqslant \langle l, \bar{u}\rangle \quad u - w \in L_0 \quad \forall \bar{u} \in L_0 \\ &\mathrm{d}\Phi_2^*(P_S\sigma; P_S\bar{\sigma}) \leqslant \langle P_S\bar{\sigma}, P_S Tu\rangle \qquad\qquad\qquad\qquad\quad \forall P_S\bar{\sigma} \in \mathscr{S} \end{aligned} \tag{9}$$

which provide the structural problem (4) in a weak form. Relation $(9)_1$ is the weak form of the equilibrium equation in which the deviatoric stress field is obtained from the generalized energy Φ_1 in terms of a deviatoric compatible strain field. Inequality $(9)_2$ provides the weak form of the spherical part of the generalized constitutive relation between spherical stresses and spherical compatible strain fields.

The spherical part of the stress can be dropped from the modified Hellinger–Reissner variational principle by imposing the fulfilment of the generalized constitutive relation between spherical stress fields and compatible spherical strain fields:

$$P_S\sigma \in \partial\Phi_2(P_S Tu) \Leftrightarrow \Phi_2(P_S Tu) + \Phi_2^*(P_S\sigma) = \langle P_S\sigma, P_S Tu\rangle.$$

Hence we have:

Proposition 3.5. Generalized total potential energy. A displacement field u is a solution of the minimization problem:

$$\min_u \Omega_4(u)$$

where:

$$\Omega_4(u) = \Phi_1(P_D Tu) + \Phi_2(P_S Tu) - \Upsilon(u) - \langle l, u\rangle$$

if and only if it is a solution of the generalized structural problem. □

4. Two limitation principles

Let us prove the conditions ensuring that solutions derived from approximate variational principles are coincident. The approximate structural models derived from the modified form of the Hu–Washizu and Hellinger–Reissner variational principles and from the generalized total potential energy are considered.

The interpolating spaces of the variables involved in the variational formulations will be left purposely unspecified in order to perform a general treatment which can be applied to a wide range of situations. It is convenient to assume the admissible displacement field $v = u - w \in L_0$ as unknown.

The approximate form of the potential Ω_2 is considered first. The finite dimensional interpolating spaces for the variables, v, $P_S\sigma$, $P_S\varepsilon$ are $\mathscr{U}_n$, $\mathscr{S}_m$, $\mathscr{D}_q$ where the pedices of the spaces represent their dimension. Without loss of generality, it can be assumed that $\mathscr{U}_n \subseteq L_0$; in fact, even if $\mathscr{U}_n \nsubseteq L_0$, only the intersection $\mathscr{U}_n \bigcap L_0$ plays the role of admissible subspace for the interpolating displacement variations.

The interpolating variables will be denoted by the apex 'bullet' and they belong to the relevant interpolating spaces:

$$v^\bullet \in \mathscr{U}_n \quad \sigma_s^\bullet \in \mathscr{S}_m \quad \varepsilon_s^\bullet \in \mathscr{D}_q. \tag{10}$$

The approximate potentials will be denoted by a superimposed hat.

The approximate expression of the three-field potential reported in the Proposition 3.3 is obtained by replacing the state variables $\{u, P_S\sigma, P_S\varepsilon\}$ of the continuum problem with their interpolating counterparts (10):

$$\hat{\Omega}_2(v^\bullet, \sigma_s^\bullet, \varepsilon_s^\bullet) = \Phi_1[P_D T(w+v^\bullet)] + \Phi_2(\varepsilon_s^\bullet) - \Upsilon(w+v^\bullet) + \langle \sigma_s^\bullet, P_S T(w+v^\bullet) - \varepsilon_s^\bullet \rangle - \langle l, w+v^\bullet \rangle. \tag{11}$$

A solution of the approximate structural problem is thus obtained as

$$\operatorname*{stat}_{\sigma_s^\bullet} \min_{v^\bullet, \varepsilon_s^\bullet} \hat{\Omega}_2(v^\bullet, \sigma_s^\bullet, \varepsilon_s^\bullet). \tag{12}$$

The approximate generalized problem related to the potential Ω_3 is then considered. The interpolating variables are:

$$v^\bullet \in \mathscr{U}_n \quad \sigma_s^\bullet \in \mathscr{S}_m. \tag{13}$$

The approximate two-field potential can be obtained from Ω_3 by replacing the state variables $\{u, P_S\sigma\}$ with their interpolating counterparts (13):

$$\hat{\Omega}_3(v^\bullet, \sigma_s^\bullet) = \Phi_1[P_D T(w+v^\bullet)] - \Phi_2^*(\sigma_s^\bullet) - \Upsilon(w+v^\bullet) + \langle \sigma_s^\bullet, P_S T(w+v^\bullet) \rangle - \langle l, w+v^\bullet \rangle \tag{14}$$

and a solution of the related structural problem is given in the form

$$\min_{v^\bullet} \max_{\sigma_s^\bullet} \hat{\Omega}_3(v^\bullet, \sigma_s^\bullet). \tag{15}$$

The limitation principle between the approximate functionals $\hat{\Omega}_2(v^\bullet, \sigma_s^\bullet, \varepsilon_s^\bullet)$ and $\hat{\Omega}_3(v^\bullet, \sigma_s^\bullet)$ can now be stated.

Denoting by $\partial\Phi_2(\mathscr{D}_q)$ the image of the restrictions of the subdifferential operator $\partial\Phi_2$ to the space $\mathscr{D}_q$, i.e. the approximate spherical part of the generalized constitutive relation, we have:

Proposition 4.1. The solutions of the two approximate problems derived from the stationarity conditions (12) and (15) coincide in terms of $\{v^\bullet, \sigma_s^\bullet\}$ if the same interpolating spaces $\mathscr{U}_n$, $\mathscr{S}_m$ are assumed and if

$$\mathscr{S}_m \subseteq \partial\Phi_2(\mathscr{D}_q). \tag{16}$$

Proof. Let us preliminarily recall that the spherical part of the generalized complementary energy Φ_2^* is the conjugate of the spherical part of the generalized energy Φ_2 (7):

$$\Phi_2^*(P_S\sigma) = \sup_{P_S\bar{\varepsilon} \in \mathscr{D}} \{\langle P_S\sigma, P_S\bar{\varepsilon}\rangle - \Phi_2(P_S\bar{\varepsilon})\} = \langle P_S\sigma, P_S\varepsilon\rangle - \Phi_2(P_S\varepsilon).$$

The 'sup' is attained at a point $\{P_S\varepsilon, P_S\sigma\}$ which fulfils the spherical part of the constitutive relation $P_S\sigma \in \partial\Phi_2(P_S\varepsilon)$.

The inclusion (16) ensures that for any approximate spherical stress $\sigma_s^\bullet \in \mathscr{S}_m$ there exists an approximate spherical strain $\varepsilon_s^\bullet \in \mathscr{D}_q$ such that the spherical part of the generalized constitutive relation is fulfilled:

$$\forall \sigma_s^\bullet \in \mathscr{S}_m \quad \exists \varepsilon_s^\bullet \in \mathscr{D}_q \mid \sigma_s^\bullet \in \partial\Phi_2(\varepsilon_s^\bullet). \tag{17}$$

The subdifferential relation appearing in (17) is the same as enforcing the following Fenchel's equality:

$$\Phi_{2a}^*(\sigma_s^\bullet) = \sup_{\bar{\varepsilon}_s^\bullet \in \mathscr{D}_q} \{\langle \sigma_s^\bullet, \bar{\varepsilon}_s^\bullet \rangle - \Phi_2(\bar{\varepsilon}_s^\bullet)\} = \langle \sigma_s^\bullet, \varepsilon_s^\bullet \rangle - \Phi_2(\varepsilon_s^\bullet). \tag{18}$$

According to (17), the 'sup' in (18) is attained at $\varepsilon_s^\bullet \in \mathscr{D}_q \subseteq \mathscr{D}$ so that the supremum can equivalently be performed over the entire space $\mathscr{D}$. The potential Φ_{2a}^* turns then out to be the restriction of Φ_2^* to $\mathscr{S}_m \subseteq \mathscr{S}$.

Substituting the relation (18) written as:

$$-\Phi_2^*(\sigma_s^\bullet) = \Phi_2(\varepsilon_s^\bullet) - \langle \sigma_s^\bullet, \varepsilon_s^\bullet \rangle$$

in the expression of the potential $\hat{\Omega}_2$, we get the potential $\hat{\Omega}_3$ and the limitation principle is proved. □

For the subsequent analysis, a limitation principle between the modified form of the Hellinger–Reissner principle and the displacement-based variational formulation is derived.

Assuming $v^\bullet \in \mathscr{U}_n$, the approximate potential $\hat{\Omega}_4$ is given by:

$$\hat{\Omega}_4(v^\bullet) = \Phi_1[P_D T(w+v^\bullet)] + \Phi_2[P_S T(w+v^\bullet)] - \Upsilon(w+v^\bullet) - \langle l, w+v^\bullet \rangle \tag{19}$$

and the related stationarity condition is:

$$\min_{v^\bullet} \hat{\Omega}_4(v^\bullet). \tag{20}$$

The following limitation principle can now be stated:

Proposition 4.2. The solutions of the approximate problems derived from the stationarity conditions (15) and (20) coincide in terms of $\{v^\bullet\}$ if the same interpolating space $\mathscr{U}_n$ is assumed and if

$$P_S T(w+\mathscr{U}_n) \subseteq \partial\Phi_2^*(\mathscr{S}_m). \tag{21}$$

Proof. The inclusion (21) ensures that for any admissible interpolating displacement there exists an interpolating spherical stress such that the spherical part of the generalized constitutive relation is fulfilled:

$$\forall v^\bullet \in \mathscr{U}_n \quad \exists \sigma_s^\bullet \in \mathscr{S}_m \mid P_S T(w+v^\bullet) \in \partial\Phi_2^*(\sigma_s^\bullet). \tag{22}$$

The subdifferential relation in (22) is the same as enforcing the Fenchel's equality:

$$\Phi_{2a}[P_S T(w+v^\bullet)] = \sup_{\bar{\sigma}_s^\bullet \in \mathscr{S}_m} \{\langle \bar{\sigma}_s^\bullet, P_S T(w+v^\bullet) \rangle - \Phi_2^*)\} = \langle \sigma_s^\bullet, P_S T(w+v^\bullet) \rangle - \Phi_2^*(\sigma_s^\bullet) \tag{23}$$

where, according to (22), the 'sup' is attained at $\sigma_s^\bullet \in \mathscr{S}_m \subseteq \mathscr{S}$. The supremum can be equivalently

Table 1
The complete set of limitation principles for the generalized elastic material

Interpolating field	Condition
$r\bullet$	$w+\mathscr{U}_n \subseteq \partial\Upsilon^*(\mathscr{R}_k)$
$\varepsilon\bullet$	$\mathscr{S}_h \subseteq \partial\Phi(\mathscr{D}_p)$
$v\bullet$	$T'\mathscr{S}_h \subseteq \partial\Upsilon(w+\mathscr{U}_n)$
$\sigma\bullet$	$T(w+\mathscr{U}_n) \subseteq \partial\Phi^*(\mathscr{S}_h)$

performed over the entire space $\mathscr{S}$ and the potential Φ_{2a} turns out to be the restriction of Φ_2 to $P_S T(w+\mathscr{U}_n)$.

Substituting the Fenchel's relation (23) in terms of the approximate variables:

$$\Phi_2[P_S T(w+v^\bullet)] = \langle \sigma_s^\bullet, P_S T(w+v^\bullet)\rangle - \Phi_2^*(\sigma_s^\bullet)$$

in the expression of the potential $\hat{\Omega}_3$, the potential $\hat{\Omega}_4$ is provided and the theorem is thus proved. □

Remark 4.1. The complete family of variational formulations for the GEM can be derived from the potential (5) and is reported in Romano (1994). The approximate discrete structural model associated with each variational principle is obtained by assuming that the four variables v, σ, ε and r belong to finite-dimensional interpolating spaces $\mathscr{U}_n$, $\mathscr{S}_h$, $\mathscr{D}_q$ and $\mathscr{R}_k$.

The limitation principles associated with the approximate variational formulations for the GEM can be proved by following the same steps performed in Propositions 4.1 and 4.2. For the sake of brevity their proofs are omitted.

Moreover it can be shown that a limitation principle relating two different formulations depends on which state variable is interpolated in one formulation and not in the other one. The complete set of limitation principles for the GEM is thus reported in Table 1 and generalizes the corresponding one provided in Alfano and Marotti de Sciarra (1996) in the context of linear elasticity. ■

In the next section the results previously obtained for the GEM are specialized to elastic and elastoplastic models.

5. B-bar method versus displacement/pressure formulation

Monotonicity of the graph $\mathscr{G}(\mathscr{E})$ for the GEM and the conditions dom $\mathscr{E} = \mathscr{D}$ and dom $\mathscr{E}^{-1} = \mathscr{S}$ ensure that the potentials Φ and Φ^* turn out to be convex, but, in general, they are not strictly convex or differentiable.

To recover the classical theory of elasticity, the graph $\mathscr{G}(\mathscr{E})$ is then assumed strictly monotone and hence the elastic energy Φ and the complementary elastic energy Φ^* are both strictly convex.

Since the strict convexity of one of them implies the differentiability of the other one and vice-versa, it follows that both potentials turn out to be differentiable. The elastic behaviour is then one-to-one and relations (1) can be written as $\sigma = \mathscr{E}\varepsilon$ and $\varepsilon = \mathscr{E}^{-1}\sigma$.

Further the assumed conservativity of the constitutive stiffness $\mathscr{E}$ yields its symmetry by virtue of the Volterra's condition (Vainberg, 1964) and the strict convexity of the elastic potentials implies that $\mathscr{E}$ and $\mathscr{E}^{-1}$ are positive definite. Accordingly, the potentials Φ and Φ^* will be given by the positive definite quadratic forms:

$$g^{\mathrm{el}}(\varepsilon) = \Phi(\varepsilon) = \tfrac{1}{2}\langle \mathscr{E}\varepsilon, \varepsilon\rangle \quad (g^{\mathrm{el}})^*(\sigma) = \Phi^*(\sigma) = \tfrac{1}{2}\langle \sigma, \mathscr{E}^{-1}\sigma\rangle. \tag{24}$$

In the context of elasticity, problems arising from the numerical treatment of the incompressibility constraint have received great attention in the literature, see e.g., Oden and Carey (1983), Gadala and Oravas (1984), Gadala (1986), Simo and Taylor (1991).

A popular approach is based on a modification of the discrete gradient operator and in the computational literature it is often referred to as the B-bar method (Zienkiewicz and Taylor, 1991). It stems from the modified form of the Hu–Washizu principle reported in Proposition 3.3 and a synthetic derivation is hereafter presented.

• *The B-bar method*. The weak form of the elastic structural model is obtained from the corresponding weak form (8) of the GEM by setting $g_D^{\mathrm{el}} = \Phi_1$ and $g_S^{\mathrm{el}} = \Phi_2$:

$$\begin{aligned}
&\langle \mathrm{d}g_D^{\mathrm{el}}(P_D Tu), P_D T\bar{u}\rangle + \langle P_S\sigma, P_S T\bar{u}\rangle = \langle l, \bar{u}\rangle \quad u - w \in L_0 && \forall \bar{u} \in L_0 \\
&\langle P_S\bar{\sigma}, P_S(Tu - \varepsilon)\rangle = 0 && \forall P_S\bar{\sigma} \in \mathscr{S} \\
&\langle \mathrm{d}g_S^{\mathrm{el}}(P_S e), P_S\bar{\varepsilon}\rangle - \langle P_S\sigma, P_S\bar{\varepsilon}\rangle = 0 && \forall P_S\bar{\varepsilon} \in \mathscr{D}.
\end{aligned} \tag{25}$$

A standard discretization of the domain $\mathscr{B}$ in finite elements is performed by means of a finite family of nonoverlapping subdomains $\mathscr{B}^e$ where $e = 1, \ldots, A$ such that $\bigcup_{e=1}^{A} \bar{\mathscr{B}}^e = \bar{\mathscr{B}}$, $\mathscr{B}^{e_1} \bigcap \mathscr{B}^{e_2} = \varnothing$ for every $e_1 \neq e_2$.

The approximate weak form of the structural problem is obtained from (25) by replacing the state variable fields with the admissible interpolating displacement field $v^\bullet$, the interpolating pressure field $\rho^\bullet$ and the interpolating volumetric strain field $\Theta^\bullet$.

Over a typical element $\mathscr{B}^e$, the displacement approximation and the related compatible strains are introduced by setting:

$$v^\bullet(x) = \sum_{i=1}^{n} N_i^u(x)\mathbf{v}_i \qquad Tv^\bullet(x) = \sum_{i=1}^{n} TN_i^u(x)\mathbf{v}_i = \sum_{i=1}^{n} \mathbf{B}_i(x)\mathbf{v}_i \tag{26}$$

where N_i^u are the standard element displacement shape functions.

As usually assumed in the literature, see e.g. Simo et al. (1985), Weiss et al. (1966), Zienkiewicz and Taylor (1991), the finite-dimensional interpolating spaces for the volumetric strain field and for the pressure field are spanned by the *same* set of shape functions $\{M_i(x)\}$ where $i = 1, \ldots, m$. Accordingly one has:

$$\Theta^\bullet(x) = \sum_{i=1}^{m} M_i(x)\Theta_i = \mathbf{M}(x)\cdot\boldsymbol{\Theta} \qquad \rho^\bullet(x) = \sum_{i=1}^{m} M_i(x)\rho_i = \mathbf{M}(x)\cdot\boldsymbol{\rho} \tag{27}$$

and no interelement continuity is enforced on $\Theta^\bullet$ and $\rho^\bullet$.

The divergence of the interpolating displacement field $v^{\bullet}$ on the finite element $\mathscr{B}^e$ is given by:

$$\operatorname{div} v^{\bullet}(x) = \mathbf{I} * \sum_{i=1}^{n} \mathbf{B}_i(x)\mathbf{v}_i = \sum_{i=1}^{n} \mathbf{b}_i(x) \cdot \mathbf{v}_i \tag{28}$$

where $\mathbf{b}_i = \mathbf{B}_i^t\mathbf{I}$, i.e. $(b_i)_k = (B_i)_{klj}I_{ij}$. Setting:

$$H^e = \int_{\mathscr{B}^e} \mathbf{M}(x) \otimes \mathbf{M}(x)\, \mathrm{d}x$$

equality $(25)_2$ provides the volumetric strain parameters $\mathbf{\Theta}$ pertaining to the element $\mathscr{B}^e$:

$$\mathbf{\Theta} = H^{e-1} \int_{\mathscr{B}^e} \mathbf{M}(x) \operatorname{div}[w(x) + v^{\bullet}(x)]\, \mathrm{d}x = (H^e)^{-1} \sum_{i=1}^{n} \int_{\mathscr{B}^e} \mathbf{M}(x) \otimes \mathbf{b}_i(x)\, \mathrm{d}x(\mathbf{w}_i + \mathbf{v}_i). \tag{29}$$

Denoting by $\hat{g}_S^{\mathrm{el}}(\Theta^{\bullet}) = g_S^{\mathrm{el}}(\varepsilon_s^{\bullet})$ the elastic energy expressed in terms of the interpolating volumetric strain field, equality $(25)_3$ provides the pressure parameters $\boldsymbol{\rho}$:

$$\boldsymbol{\rho} = (H^e)^{-1} \int_{\mathscr{B}^e} \mathrm{d}\hat{g}_S^{\mathrm{el}}(\Theta^{\bullet})\mathbf{M}(x)\, \mathrm{d}x. \tag{30}$$

In isotropic linear elasticity, the spherical part of the elastic energy is given by $\hat{g}_S^{\mathrm{el}}(\Theta) = K/2\langle\Theta, \Theta\rangle$, so that $\mathrm{d}\hat{g}_S^{\mathrm{el}}(\Theta^{\bullet}) = K\mathbf{M} \cdot \mathbf{\Theta}$ and (30) becomes

$$\boldsymbol{\rho} = K\mathbf{\Theta}. \tag{31}$$

The discrete equilibrium equation is obtained by substituting the interpolating fields expressed in terms of parameters (29) and (31) in the relevant fields appearing in $(25)_1$ to get:

$$\sum_{i,j=1}^{n} \int_{\mathscr{B}^e} \mathscr{E}\bar{\mathbf{B}}_i(\mathbf{w}_i + \mathbf{v}_i) * \bar{\mathbf{B}}_j\bar{\mathbf{v}}_j\, \mathrm{d}x = \sum_{j=1}^{n} \mathbf{l}_j \cdot \bar{\mathbf{v}}_j \quad \forall \bar{\mathbf{v}}_j \in \Re^n. \tag{32}$$

The modified discrete gradient operator $\bar{\mathbf{B}}_i = \mathbf{B}_{Di} + \bar{\mathbf{B}}_{Si}$ represents the B-bar matrix and differs from the usual matrix $\mathbf{B}_i$ in the symmetric part $\bar{\mathbf{B}}_{Si}$:

$$\bar{\mathbf{B}}_{Si}\mathbf{v}_i = \frac{1}{3}(\bar{\mathbf{b}}_i \cdot \mathbf{v}_i)\mathbf{I} \quad \text{where } \bar{\mathbf{b}}_i^t(x) = \mathbf{M}^t(x)(H^e)^{-1} \int_{\mathscr{B}^e} \mathbf{M}(x) \otimes \mathbf{b}_i(x)\, \mathrm{d}x.$$

The formulation (32) reduces to the B-bar method reported in Zienkiewicz and Taylor (1991) by replacing the tensorial notation with a vectorial one and noting the following identities: $H^e = \mathbf{E}_Z$, $\int_{\mathrm{B}e} \mathbf{b} \otimes \mathbf{M}\, \mathrm{d}x = \mathbf{C}_Z$ and $\bar{\mathbf{B}}_S = 1/3\mathbf{m}_Z\mathbf{N}_Z^p\mathbf{E}_Z^{-1}\mathbf{C}_Z^t$ where $\mathbf{m}_Z$ depends on the problem at hand; for a three-dimensional model it is $\mathbf{m}_Z = [1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0]^t$.

• *The displacement/pressure formulation*. Mixed displacement/pressure approximations, see Nagtegaal et al. (1974), Key (1969), can be obtained from the modified version of the Hellinger–Reissner principle presented in the previous section.

The weak form of the elastic structural model is obtained from equations (9):

$$\begin{array}{ll}\langle \mathrm{d}g_D^{\mathrm{el}}(P_D T u), P_D T\bar{u}\rangle + \langle P_S\sigma, P_S T\bar{u}\rangle = \langle l, \bar{u}\rangle \quad u-w\in L_0 & \forall \bar{u}\in L_0 \\ \langle P_S\bar{\sigma}, \mathrm{d}(g_S^{\mathrm{el}})^*(P_S\sigma)\rangle - \langle P_S\bar{\sigma}, P_S T u\rangle = 0 & \forall P_S\bar{\sigma}\in\mathscr{S}.\end{array} \tag{33}$$

The displacement approximation over a typical finite element $\mathscr{B}^e$ is defined according to (26). The finite-dimensional interpolating space for the pressure field is spanned by the set of shape functions $\{M_i(x)\}$ where $i = 1, \ldots, m$. The interpolating pressure is given by $(27)_2$ and is constructed for each element individually so that it is not continuous across element boundary.

In the case of isotropic linear elasticity, the weak eqns (33) become:

$$\begin{array}{ll}\sum_{i=1}^{n}\int_{\mathscr{B}^e}\mathbf{M}\otimes\mathbf{b}_i\,\mathrm{d}x(\mathbf{w}_i+\mathbf{v}_i)\cdot\bar{\boldsymbol{\rho}} - \int_{\mathscr{B}^e}\frac{1}{K}\mathbf{M}\otimes\mathbf{M}\,\mathrm{d}x\boldsymbol{\rho}\cdot\bar{\boldsymbol{\rho}} = 0 & \forall\bar{\boldsymbol{\rho}}\in\Re^m \\ \sum_{i,j=1}^{n}\int_{\mathscr{B}^e}2G\mathbf{B}_{Di}(\mathbf{w}_i+\mathbf{v}_i)\cdot\mathbf{B}_{Dj}\bar{\mathbf{v}}_j\,\mathrm{d}x + \sum_{j=1}^{n}\int_{\mathscr{B}^e}\mathbf{b}_j\otimes\mathbf{M}\,\mathrm{d}x\boldsymbol{\rho}\cdot\bar{\mathbf{v}}_j = \sum_{j=1}^{n}\mathbf{l}_j\cdot\bar{\mathbf{v}}_j & \forall\bar{\mathbf{v}}_j\in\Re^n\end{array}. \tag{34}$$

It is immediate to show that eqns (34) can be specialized to the ones reported in Zienkiewicz and Taylor (1991).

5.1. The limitation principle

Denoting by $\mathscr{E}_S$ the spherical part of the material elastic stiffness $\mathscr{E}$, Proposition 4.1 becomes:

Proposition 5.1. The solutions of the two approximate elastic problems derived from the weak forms (25) and (33) corresponding to the B-bar method and to the displacement/pressure formulation coincide in terms of $\{v^\bullet, \sigma_s^\bullet\}$ if the same interpolating spaces $\mathscr{U}_n$, $\mathscr{S}_m$ are considered and if

$$\mathscr{S}_m \subseteq \mathscr{E}_S\mathscr{D}_q. \tag{35}$$ □

In isotropic elasticity, the spherical part of the constitutive relation is a scalar condition so that the limitation principle above holds true if the interpolating space for the pressure field is contained in the space of the interpolating volumetric strain field, i.e. $\mathscr{S}_m \subseteq \mathscr{D}_q$. In the literature, see e.g., Simo et al. (1985), Weiss et al. (1996), Zienkiewicz and Taylor (1991), as well as in computer codes, the interpolating spaces for the pressure field and for the volumetric strain field turn out to be coincident since the same set of shape functions $\{M_i\}$ is considered. The limitation principle stated in the Proposition 5.1 is fulfilled and the two-field displacement/pressure approximation can be successfully adopted instead of using the B-bar formulation.

5.1.1. A numerical simulation

Two numerical examples are provided to show the effectiveness of the limitation principle. The former is based on the B-bar formulation (32) and the latter deals with the mixed displacement/pressure method (34).

Over a single quadrilateral finite element, a bilinear isoparametric interpolation for the displacement field is employed together with constant pressure for the displacement/pressure approach, and constant pressure and constant volumetric strain for the B-bar method.

Remark 5.1. A bilinear isoparametric interpolation for the displacement field with constant pressure locks, i.e. it shows a loss of accuracy in the computed response as incompressibility is enforced. The degree of locking is quite small if compared with the response of an eight-node displacement element with three or four pressure parameters or with the nine-node displacement element with four pressure parameters (Sussman and Bathe, 1987). A pressure filter can be devised in order to adjust the approximate pressure field so that the Babuska–Brezzi condition (Babuska, 1973; Brezzi, 1974; Brezzi and Fortin, 1991; Pastor et al., 1997) is satisfied independently of the mesh, see e.g., Oden and Carey (1983), Pitkaranta and Stenberg (1984), Oden and Kikuchi (1982). ■

The elastic distorted tapared plate shown in Fig. 1 is considered where (a), (b) and (c) represent the three considered discretizations. The Young's modulus is $E = 70$ KN/mm^2 and the three different values of the Poisson's ratio are $\nu_1 = 0.3$, $\nu_2 = 0.4$, $\nu_3 = 0.4999$. The load condition is represented by a tangential vertical load $q = 10$ N/mm^2.

In Fig. 2 the displacements of the top corner node are reported; they are evaluated for the different meshes of Fig. 1 and for the three values of the Poisson's ratio. The pressure p is addressed in Figs 3–5 for the elements belonging to the bottom edge of the meshes shown in Fig. 1. Actually, displacements and stresses computed by the B-bar formulation and by the displacement/pressure turn out to be coincident in accordance with the limitation principle.

6. Displacement/pressure formulation versus displacement method

In the context of elasticity and elastoplasticity, modified versions of the Hellinger–Reissner variational principle have often been used by constructing displacement/pressure mixed approximations (Nagtegaal, 1974; Key, 1969). The equivalence of mixed methods with discontinuous pressure approximations and displacement methods employing selective reduced integration techniques has been investigated by many authors, e.g., Oden and Carey (1983), Oden and Kikuchi (1982), Hughes (1977), Malkus and Hughes (1978).

A general condition for the equivalence between the mixed displacement/pressure formulation and the displacement approach can be deduced from the Proposition 4.2 and is reported in the next

Proposition 6.1. The solutions of the approximate linear elastic problems derived from the weak form of the displacement/pressure approach (33) and from the displacement formulation coincide in terms of $\{v^\bullet\}$ if the some interpolating space $\mathscr{U}_n$ is considered and if

$$P_S T \mathscr{U}_n \subseteq \mathscr{E}_S^{-1} \mathscr{S}_m. \qquad (36) \quad \square$$

This limitation principle provides a theoretical basis to the numerical argumentation reported in Sussman and Bathe (1987) to find a condition for the coincidence of the finite element solutions in terms of displacements between the displacement/pressure approach and the displacement formulation.

As an example, a plane strain four-node quadrilateral element is considered and its local numbering is reported in Fig. 6. Only one finite element $\mathscr{B}^e$ is addressed since the volumetric strain field and the pressure field are discontinuous across adjacent elements.

It will be shown hereafter that the limitation principle is fulfilled if the parameters for the

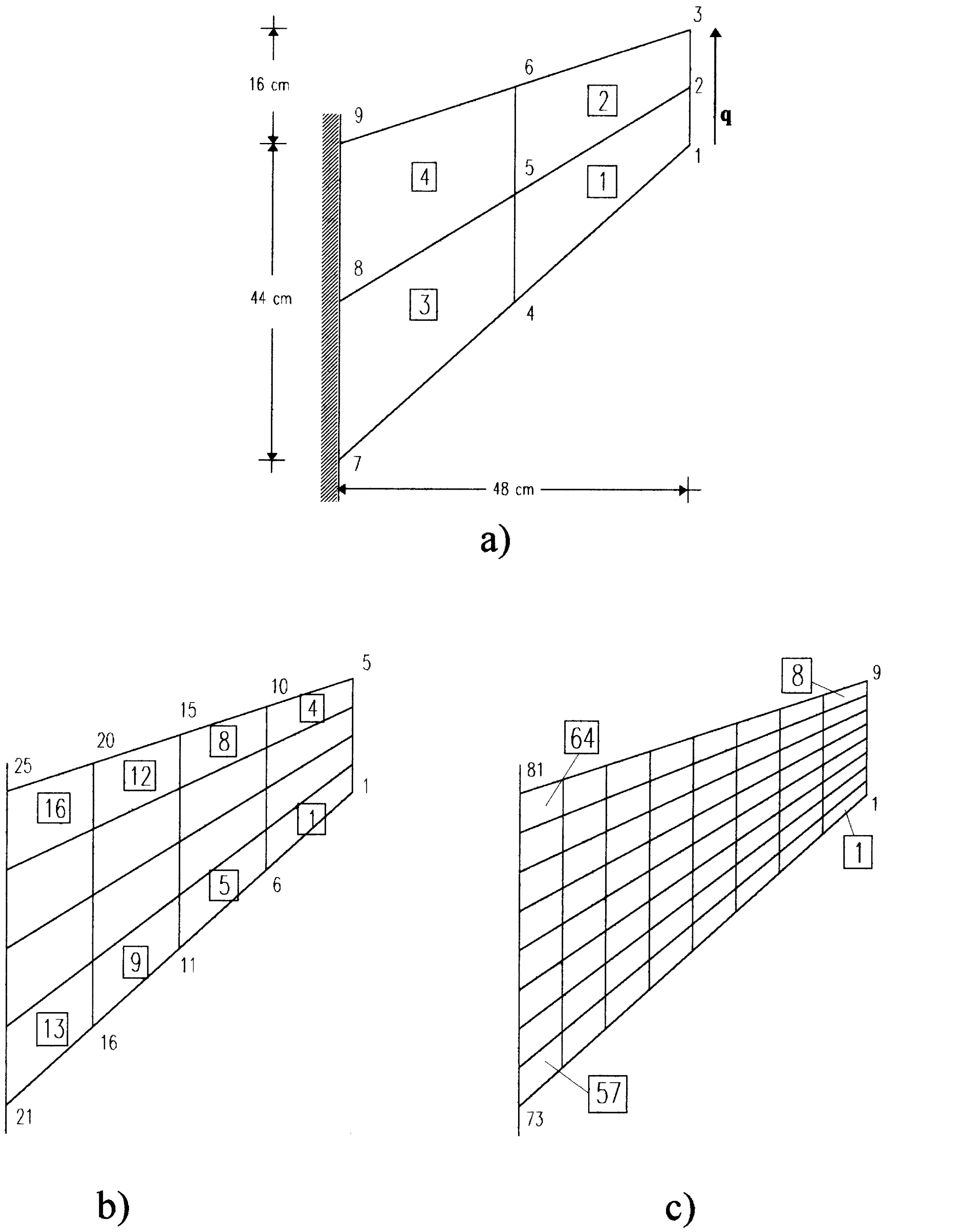

Fig. 1. The discretization of the tapered plate. (a) 2 × 2 side elements, (b) 4 × 4 side elements, (c) 8 × 8 side elements.

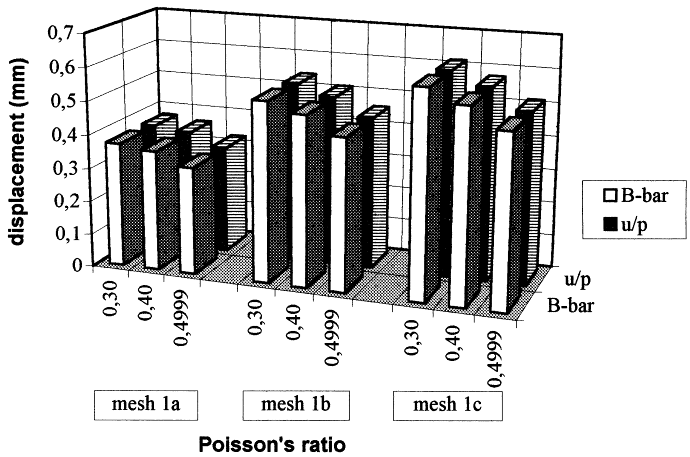

Fig. 2. The displacements of the top corner node of the tapared plate obtained from the B-bar method and the u/p formulation under the validity of the limitation principle for the three values of the Poisson's ratio.

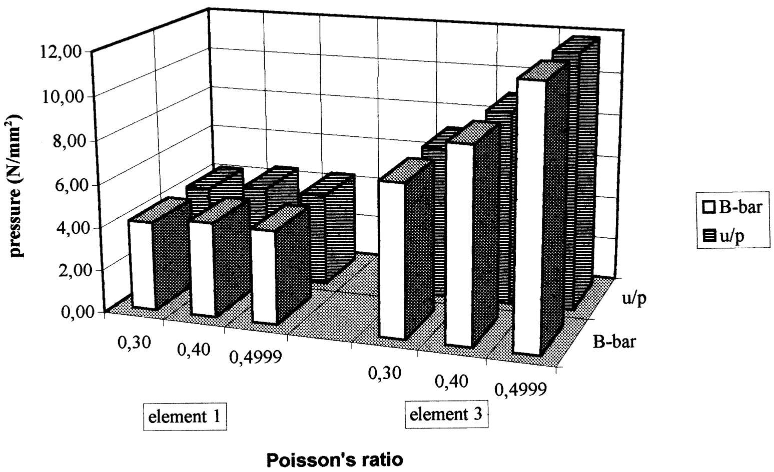

Fig. 3. The pressure for the elements of the bottom edge of the mesh of Fig. 1a is obtained from the B-bar method and the u/p formulation under the validity of the limitation principle.

interpolating pressure field are chosen at the four Gauss points of the elements. Hence the solutions of the displacement/pressure mixed approximation coincide with the solutions of the displacement method in terms of the displacement field. Different kinds of elements can be tested following the same procedure.

In terms of local coordinates $\{\xi, \eta\}$ the shape functions for each component of the displacement are the continuous functions:

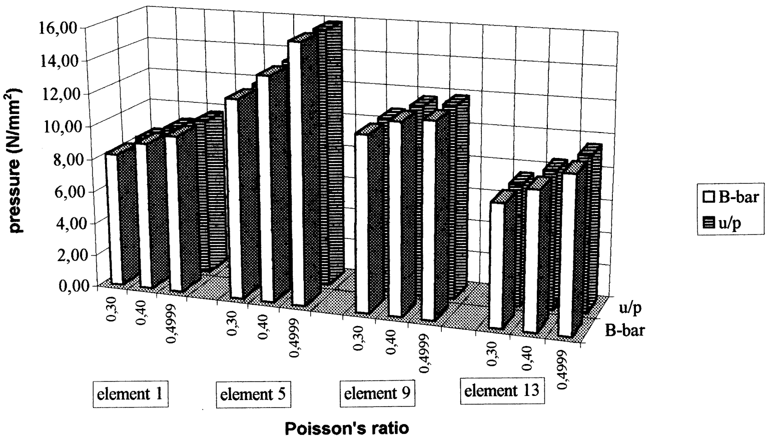

Fig. 4. The pressure for the elements of the bottom edge of the mesh of Fig. 1b is obtained from the B-bar method and the u/p formulation under the validity of the limitation principle.

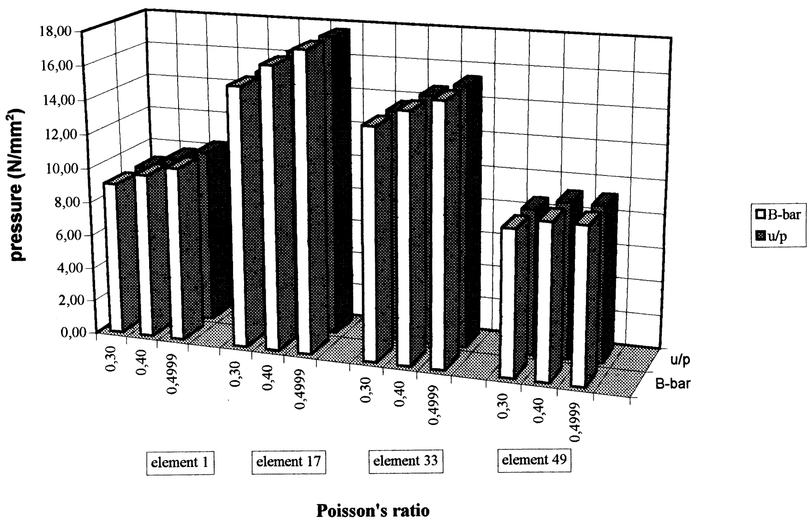

Fig. 5. The pressure for the elements 1, 17, 33 and 49 belonging to the bottom edge of the mesh of Fig. 1c is obtained from the B-bar method and the u/p formulation under the validity of the limitation principle.

$$\psi_i(\xi,\eta) = \frac{1}{4}(1+\alpha_i\xi)(1+\beta_i\eta) \begin{cases} \alpha_1 = -1 \\ \alpha_2 = 1 \\ \alpha_3 = 1 \\ \alpha_4 = -1 \end{cases} \begin{cases} \beta_1 = -1 \\ \beta_2 = -1 \\ \beta_3 = 1 \\ \beta_4 = 1 \end{cases}$$

The numerical vector $\mathbf{u}$ of the displacement parameters for the element $\mathscr{B}^e$ is ordered in the standard manner $\mathbf{u} = [\mathbf{u}_\xi \quad \mathbf{u}_\eta]^t$ where $\mathbf{u}_\xi = [u_1, \ldots, u_4]^t$ and $\mathbf{u}_\eta = [v_1, \ldots, v_4]^t$.

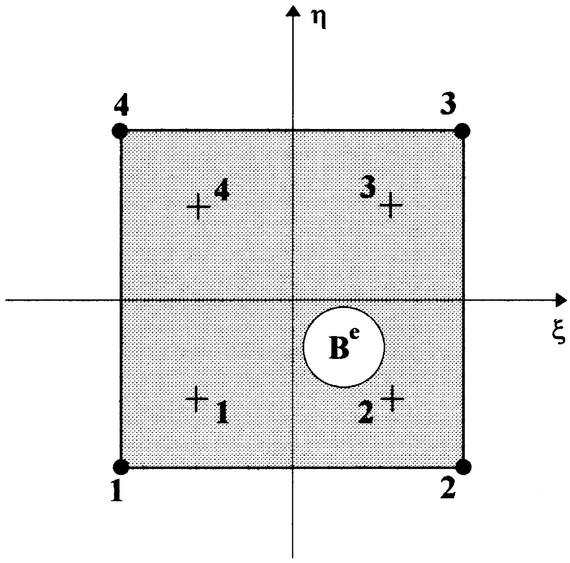

Fig. 6. The four node master element $\mathscr{B}^e$, $\bullet$ = interpolation nodes for displacements; $+$ = interpolation nodes for the pressure which coincide with the Gauss points.

The pressure field is interpolated in terms of four parameters; each of them is the value of the pressure field at one of the four Gauss points of $\mathscr{B}^e$. The shape functions are:

$$M_i(\xi,\eta) = \frac{1}{4}\left(1+\alpha_i\frac{\xi}{\xi_g}\right)\left(1+\beta_i\frac{\eta}{\eta_g}\right) \tag{37}$$

where $\xi_g = \eta_g = 1/\sqrt{3}$.

The expression of the spherical part of the strain field can be evaluated from the displacement vector $\mathbf{u}$ by means of the spherical part $\mathbf{B}_S$ of the discrete gradient operator $\mathbf{B}$ as follows:

$$\varepsilon_s^\bullet(\xi,\eta) = \begin{bmatrix} \varepsilon_\xi \\ \varepsilon_\eta \\ \varepsilon_\zeta \\ \gamma_{\xi\eta} \end{bmatrix} = \mathbf{B}_S(\xi,\eta)\mathbf{u}.$$

The matrix $\mathbf{B}_S$, expressed in terms of the shape functions ψ_i, is given by:

$$\mathbf{B}_S = \frac{1}{3}\begin{bmatrix} \psi_1^\xi & \psi_1^\eta & \psi_2^\xi & \cdots & \cdots & \cdots & \cdots & \psi_4^\eta \\ \psi_1^\xi & \psi_1^\eta & \psi_2^\xi & \cdots & \cdots & \cdots & \cdots & \psi_4^\eta \\ \psi_1^\xi & \psi_1^\eta & \psi_2^\xi & \cdots & \cdots & \cdots & \cdots & \psi_4^\eta \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

where $\psi_i^\bullet$ represents the derivative of the function ψ_i with respect to $\bullet$. Accordingly the sub-space

$P_S T\mathscr{U}_n$ is generated by the functions $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5\}$ which represent the four linearly independent columns of the matrix $\mathbf{B}_S$.

The spherical part of the strain field in the element $\mathscr{B}^e$ is now evaluated from the pressure field $\rho(\xi, \eta) = \mathbf{M}(\xi, \eta) \cdot \boldsymbol{\rho}$ and is given by:

$$\varepsilon_s^{\bullet}(\xi, \eta) = \begin{bmatrix} \varepsilon_\xi \\ \varepsilon_\eta \\ \varepsilon_\zeta \\ \gamma_{\xi\eta} \end{bmatrix} = \frac{1}{3K} \rho(\xi, \eta) \mathbf{m} = \frac{1}{3K} [\mathbf{m} \otimes \mathbf{M}(\xi, \eta)] \boldsymbol{\rho}$$

where $\mathbf{m} = [1 \quad 1 \quad 1 \quad 0]'$. The rows of the matrix $\mathbf{m} \otimes \mathbf{M}(\xi, \eta)$ are given by the assumed shape functions of the pressure field (37) and the columns of the matrix $\mathbf{m} \otimes \mathbf{M}$ are denoted by $\hat{\mathbf{h}}_i$ where $i = 1, \ldots, 4$.

The inclusion (36) can then be checked on the basis of a theorem of linear algebra (Luenberger, 1973): a set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p\}$ belonging to a linear space $\mathscr{A}$ endowed with a scalar product $(\cdot, \cdot)$ is linearly independent if and only if the Gramm matrix $\mathbf{G}$ defined as $G_{ij} = (\mathbf{a}_i, \mathbf{a}_j)$ is not singular.

Accordingly the first step consists in evaluating the Gramm matrix $\mathbf{G}^{\mathrm{h}}$ related to the functions $\hat{\mathbf{h}}_i$. Since we have:

$$\det \begin{bmatrix} (\hat{\mathbf{h}}_1, \hat{\mathbf{h}}_1) & (\hat{\mathbf{h}}_1, \hat{\mathbf{h}}_2) & (\hat{\mathbf{h}}_1, \hat{\mathbf{h}}_3) & (\hat{\mathbf{h}}_1, \hat{\mathbf{h}}_4) \\ (\hat{\mathbf{h}}_2, \hat{\mathbf{h}}_1) & \ldots & \ldots & \ldots \\ (\hat{\mathbf{h}}_3, \hat{\mathbf{h}}_1) & \ldots & \ldots & \ldots \\ (\hat{\mathbf{h}}_4, \hat{\mathbf{h}}_1) & \ldots & \ldots & (\hat{\mathbf{h}}_4, \hat{\mathbf{h}}_4) \end{bmatrix} = 3 \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \neq 0$$

it can be deduced that the functions $\{\hat{\mathbf{h}}_i\}$ provide a basis for the space $\mathscr{E}_S^{-1} \mathscr{S}_m$.

The second step consists in adding the functions $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5\}$ to the set $\{\hat{\mathbf{h}}_1, \hat{\mathbf{h}}_2, \hat{\mathbf{h}}_3, \hat{\mathbf{h}}_4\}$ and in evaluating the relevant Gramm matrix $\mathbf{G}^{\mathrm{he}_i}$. A direct calculation shows that:

$$\det \begin{bmatrix} 3 & 0 & 0 & 0 & (\hat{\mathbf{h}}_1, \hat{\mathbf{e}}_j) \\ 0 & 3 & 0 & 0 & (\hat{\mathbf{h}}_2, \hat{\mathbf{e}}_j) \\ 0 & 0 & 3 & 0 & (\hat{\mathbf{h}}_3, \hat{\mathbf{e}}_j) \\ 0 & 0 & 0 & 3 & (\hat{\mathbf{h}}_4, \hat{\mathbf{e}}_j) \\ (\hat{\mathbf{h}}_1, \hat{\mathbf{e}}_j) & (\hat{\mathbf{h}}_2, \hat{\mathbf{e}}_j) & (\hat{\mathbf{h}}_3, \hat{\mathbf{e}}_j) & (\hat{\mathbf{h}}_4, \hat{\mathbf{e}}_j) & (\hat{\mathbf{e}}_j, \hat{\mathbf{e}}_j) \end{bmatrix} = 0$$

for any $\hat{\mathbf{e}}_j$ where $j = 1, 2, 4, 5$. We can then conclude that the vectors $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_4, \hat{\mathbf{e}}_5\}$ spanning the subspace $P_S T\mathscr{U}_n$ are linearly dependent from $\{\hat{\mathbf{h}}_1, \hat{\mathbf{h}}_2, \hat{\mathbf{h}}_3, \hat{\mathbf{h}}_4\}$ so that the inclusion (36) is fulfilled and the solutions in terms of displacements obtained from the mixed displacement/pressure approach and from the displacement one coincide.

6.1. A numerical simulation

A finite element analysis of the tapared plate reported in Fig. 1 is performed by adopting the mixed displacement/pressure method and the displacement-based approach. The quadrilateral

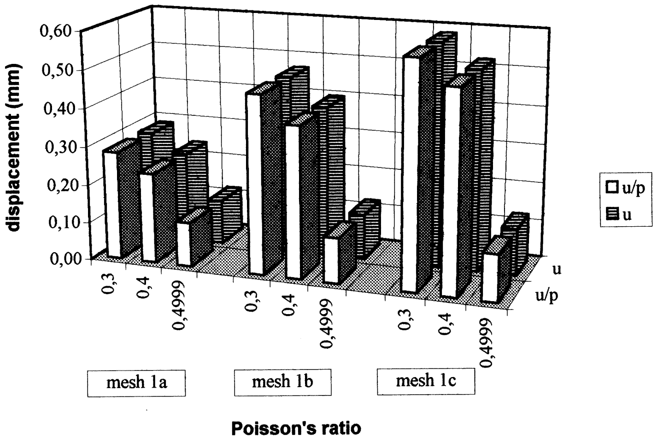

Fig. 7. The displacements of the top corner node of the tapared plate obtained from the u/p formulation and the displacement method under the validity of the limitation principle for the three values of the Poisson's ratio.

element analysed in the previous section is considered. The Young's modulus is $E = 70$ KN/mm^2 and the three different values of the Poisson's ratio are $v_1 = 0.3$, $v_2 = 0.4$, $v_3 = 0.4999$. The load condition is represented by a tangential vertical load $q = 10$ N/mm^2. The displacements of the top corner node evaluated for the different meshes of Fig. 1 and for the three values of the Poisson's ratio are reported in Fig. 7. The computed displacements turn out to be coincident in agreement with the limitation principle.

7. Elastoplastic behaviour

A rate-independent standard elastoplastic model with mixed hardening, representing the Generalized Standard Material (GSM) initially proposed by Halphen and Nguyen (1975), is now derived from the GEM.

The kinematic and isotropic hardening is introduced by means of two static internal variable fields $\chi_1 \in \mathscr{X}'$ and $\chi_2 \in \mathfrak{R}$. The elastic domain $C \subseteq \mathscr{S} \times \mathscr{X}' \times \mathfrak{R}$ is defined in terms of stress fields and static internal variable fields so that the back-stress χ_1 provides the shift of the elastic domain C and χ_2 gives a measure of the expansion of C. The kinematic internal variable fields $\alpha_1 \in \mathscr{X}$ and $\alpha_2 \in \mathfrak{R}$ are dual of χ_1 and χ_2.

The kinematic and static internal variables can be grouped into two vectors $\alpha = [\alpha_1 \quad \alpha_2]'$ and $\chi = [\chi_1 \quad \chi_2]'$. The generalized total, elastic and plastic strain fields, denoted by $\underset{\sim}{e}$, $\underset{\sim}{p}$ and $\underset{\sim}{\varepsilon}$, and the generalized stress field $\underset{\sim}{\sigma}$ can be defined in the form:

$$\underset{\sim}{e} = \begin{bmatrix} e \\ \alpha \end{bmatrix} \quad \underset{\sim}{p} = \begin{bmatrix} p \\ -\alpha \end{bmatrix} \quad \underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} \quad \underset{\sim}{\sigma} = \begin{bmatrix} \sigma \\ \chi \end{bmatrix}.$$

The kinematic operator $\underset{\sim}{T}$ *is given by* $\underset{\sim}{T} = [T \quad 0 \quad 0]'$ and the scalar product between generalized variables is defined by $\prec \underset{\sim}{\sigma}, \underset{\sim}{e} \succ = \langle \sigma, e \rangle + \langle \chi_1, \alpha_1 \rangle + \langle \chi_2, \alpha_2 \rangle$.

It is well known that the computational solution of an elastoplastic problem is obtained by solving a sequence of finite-step problems. According to a fully implicit integration scheme (Euler backward difference) (Ortiz and Simo, 1986), each finite-step problem amounts to evaluating the unknown variables $(\bullet)$ at the time t_{n+1} starting from the known values $(\bullet)_0$ at the beginning of the time step t_n.

A generalized complementary relation is considered, in which a limitation is imposed on the range of the stress field by means of the convex indicator $\sqcup_C$ of the elastic domain. Denoting by g^* the complementary free energy (Lubliner, 1990), we get

$$\Phi^*(\underset{\sim}{\sigma}) = g^*(\underset{\sim}{\sigma}) + \sqcup_C(\underset{\sim}{\sigma}) + \prec \underset{\sim}{\sigma}, \underset{\sim}{p}_0 \succ. \tag{38}$$

The generalized energy Φ for the GEM is provided by the conjugate of (38). Recalling that the conjugate of the sum of convex functionals is given by the inf-convolution of the conjugate of the addends (Hiriart-Urruty and Lemaréchal, 1993):

$$g(\underset{\sim}{e}) = \sup\{\prec \underset{\sim}{\sigma}, \underset{\sim}{e} \succ - g^*(\underset{\sim}{\sigma})\}$$

$$D(\underset{\sim}{p} - \underset{\sim}{p}_0) = \sup\{\prec \underset{\sim}{\sigma}, \underset{\sim}{p} - \underset{\sim}{p}_0 \succ - \sqcup_C(\underset{\sim}{\sigma})\}$$

the following explicit expression is obtained:

$$\Phi(\underset{\sim}{\varepsilon}) = \inf\{g(\bar{\underset{\sim}{e}}) + D(\bar{\underset{\sim}{p}} - \underset{\sim}{p}_0) \mid \bar{\underset{\sim}{e}} + \bar{\underset{\sim}{p}} = \underset{\sim}{\varepsilon}\} = g(\underset{\sim}{e}) + D(\underset{\sim}{p} - \underset{\sim}{p}_0). \tag{39}$$

The infimum is attained in correspondence of a point $\{\underset{\sim}{e}, \underset{\sim}{p}\}$ such that the following conditions are fulfilled:

$$\underset{\sim}{e} \in \partial g^*(\underset{\sim}{\sigma}) \quad (\underset{\sim}{p} - \underset{\sim}{p}_0) \in N_C(\underset{\sim}{\sigma}) \quad \underset{\sim}{e} + \underset{\sim}{p} = \underset{\sim}{\varepsilon} \tag{40}$$

where $N_C(\underset{\sim}{\sigma}) = \partial \sqcup_C(\underset{\sim}{\sigma})$ is the normal cone to the elastic domain C at the point $\underset{\sim}{\sigma} \in \partial\Phi(\underset{\sim}{\varepsilon})$.

Relations (40) provide the elastoplastic constitutive model for the GSM; in fact $(40)_1$ yields the elastic and hardening relations, $(40)_2$ provides the finite-step flow rule and $(40)_3$ represents the strain additivity.

In many problems of structural interest, the elastic domain is defined in terms of deviatoric stress fields so that the dissipation can be expressed in the form:

$$\begin{aligned} D(\underset{\sim}{p} - \underset{\sim}{p}_0) &= \sup\{\prec \underset{\sim}{P}_D \underset{\sim}{\sigma}, \underset{\sim}{P}_D(\underset{\sim}{p} - \underset{\sim}{p}_0) \succ \mid \underset{\sim}{P}_D \underset{\sim}{\sigma} \in C\} \\ &\quad + \sup\{\prec \underset{\sim}{P}_S \underset{\sim}{\sigma}, P_S(\underset{\sim}{p} - \underset{\sim}{p}_0) \succ \mid \underset{\sim}{P}_S \underset{\sim}{\sigma} \in \mathscr{S} \times \mathscr{X}' \times \Re\} \\ &= D_D[\underset{\sim}{P}_D(\underset{\sim}{p} - \underset{\sim}{p}_0)] + \sup\{\prec \underset{\sim}{P}_S \underset{\sim}{\sigma}, \underset{\sim}{P}_S(\underset{\sim}{p} - \underset{\sim}{p}_0) \succ \mid \underset{\sim}{P}_S \underset{\sim}{\sigma} \in \mathscr{S} \times \mathscr{X}' \times \Re\} \end{aligned}$$

where $\underset{\sim}{P}_D = \mathrm{diag}\,[P_D \quad P_D \quad 1]$ and $\underset{\sim}{P}_S = \mathrm{diag}\,[P_S \quad P_S \quad 0]$ are projector operators.

Moreover the free energy $g(\underset{\sim}{e})$ can be additively decomposed as the sum of the elastic energy, given by a convex functional $g^{\mathrm{el}}(e)$, and of a convex hardening potential $g^{\mathrm{h}}(\alpha)$ which describes the role of the internal variables in the hardening processes. In terms of spherical and deviatoric strain fields, we have:

$$g(\underset{\sim}{e}) = g_S(\underset{\sim}{P}_S \underset{\sim}{e}) + g_D(\underset{\sim}{P}_D \underset{\sim}{e}) \quad \text{where} \begin{cases} g_S(\underset{\sim}{P}_S \underset{\sim}{e}) = g_S^{\mathrm{el}}(P_S e) + g_S^{\mathrm{h}}(P_S \alpha_1) \\ g_D(\underset{\sim}{P}_D \underset{\sim}{e}) = g_D^{\mathrm{el}}(P_D e) + g_D^{\mathrm{h}}(P_D \alpha_1, \alpha_2). \end{cases} \tag{41}$$

The modified form of the Hu–Washizu principle reported in Proposition 3.3 is thus specialized to the GSM in the next:

Proposition 7.1. Elastoplastic Hu–Washizu principle. A vector $(u, \underset{\sim}{P}_S\underset{\sim}{\sigma}, \underset{\sim}{P}_D\underset{\sim}{p}, \underset{\sim}{P}_S\underset{\sim}{e})$ is a solution of the minimization problem:

$$\operatorname*{stat}_{\underset{\sim}{P}_S\underset{\sim}{\sigma}} \min_{u,\underset{\sim}{P}_S\underset{\sim}{\sigma},\underset{\sim}{P}_D\underset{\sim}{p},\underset{\sim}{P}_S\underset{\sim}{e}} \Sigma_1(u, \underset{\sim}{P}_S\underset{\sim}{\sigma}, \underset{\sim}{P}_D\underset{\sim}{p}, \underset{\sim}{P}_S\underset{\sim}{e}) \tag{42}$$

where:

$$\Sigma_1(u, \underset{\sim}{P}_S\underset{\sim}{\sigma}, \underset{\sim}{P}_D\underset{\sim}{p}, \underset{\sim}{P}_S\underset{\sim}{e}) = g_S(\underset{\sim}{P}_S\underset{\sim}{e}) + g_D[\underset{\sim}{P}_D(\underset{\sim}{T}u-\underset{\sim}{p})] + D_D[\underset{\sim}{P}_D(\underset{\sim}{p}-\underset{\sim}{p}_0)] - \Upsilon(u)$$
$$+ \prec \underset{\sim}{P}_S\underset{\sim}{\sigma}, \underset{\sim}{P}_S(\underset{\sim}{T}u-\underset{\sim}{e}-\underset{\sim}{p}_0)\succ - \langle l, u\rangle$$

if and only if it is a solution of the elastoplastic structural problem. □

A mixed finite element approximation can be derived from this variational principle which represents the elastoplastic version of the B-bar method, see Simo et al. (1985).

The specialization of the modified Hellinger–Reissner principle reported in Proposition 3.4 is:

Proposition 7.2. Elastoplastic Hellinger–Reissner principle. A vector $(u, \underset{\sim}{P}_S\underset{\sim}{\sigma}, \underset{\sim}{p})$ is a solution of the minimization problem:

$$\operatorname*{stat}_{\underset{\sim}{P}_S\underset{\sim}{\sigma}} \min_{u,\underset{\sim}{P}_D\underset{\sim}{p}} \Sigma_2(u, \underset{\sim}{P}_S\underset{\sim}{\sigma}, \underset{\sim}{p}) \tag{43}$$

where:

$$\Sigma_2(u, \underset{\sim}{P}_S\underset{\sim}{\sigma}, \underset{\sim}{p}) = g_S[\underset{\sim}{P}_S(\underset{\sim}{T}u-\underset{\sim}{p})] + g_D[\underset{\sim}{P}_D(\underset{\sim}{T}u-\underset{\sim}{p})] + D_D[\underset{\sim}{P}_D(\underset{\sim}{p}-\underset{\sim}{p}_0)] - \Upsilon(u)$$
$$+ \prec \underset{\sim}{P}_S\underset{\sim}{\sigma}, \underset{\sim}{P}_S(\underset{\sim}{p}-\underset{\sim}{p}_0)\succ - \langle l, u\rangle$$

if and only if it is a solution of the elastoplastic structural problem. □

It is worth noting that the spherical plastic strain field p and the spherical internal variable fields χ_1 and α_1 do not enter into the elastoplastic formulation since they are purely deviatoric fields as shown in Appendix II so that, from (41), we have $g_S(\underset{\sim}{P}_S\underset{\sim}{e}) = g_S^{\mathrm{el}}(P_S e)$. Accordingly the free energy (39) can be rewritten as:

$$\Phi(\underset{\sim}{\varepsilon}) = g_S^{\mathrm{el}}(P_S e) + g_D(\underset{\sim}{P}_D\underset{\sim}{e}) + D_D[\underset{\sim}{P}_D(\underset{\sim}{p}-\underset{\sim}{p}_0)]$$

and it turns out to be $\Phi_S(\underset{\sim}{P}_S\underset{\sim}{\varepsilon}) = g_S^{\mathrm{el}}(P_S e)$.

If the following interpolating spaces

$$v^\bullet \in \mathscr{U}_n \quad P_S\sigma_s^\bullet \in \mathscr{S}_m \quad P_S e^\bullet \in \mathscr{D}_q \quad P_D p^\bullet \in \mathscr{D}_g \quad P_D\alpha_1^\bullet \in \mathscr{X}_{1b} \quad \alpha_2^\bullet \in \mathscr{X}_{2c} \tag{44}$$

are assumed for the variables involved in the potentials Σ_1 and Σ_2, the limitation principle reported in the Proposition 4.1 can be specialized to the GSM to get:

Proposition 7.3. The solutions of the two approximate elastoplastic problems derived from the weak forms associated with (42) and (43) coincide in terms of $\{v^\bullet, P_S\sigma_s^\bullet, P_D p^\bullet, P_D\alpha_1^\bullet, \alpha_2^\bullet\}$ if the same interpolating spaces $\mathscr{U}_n$, $\mathscr{S}_m$, $\mathscr{D}_g$, $\mathscr{X}_{1b}$, $\mathscr{X}_{2c}$ are assumed and

$$\mathscr{S}_m \subseteq \partial g_S^{\mathrm{el}}(\mathscr{D}_q). \tag{45}$$ □

The same interpolating subspaces for spherical elastic strains and stresses have been adopted in the elastoplastic finite element analysis based on the Hu–Washizu principle performed in Simo et al. (1985). According to the above limitation principle, solutions in terms of displacements, spherical stresses and plastic strains derived from the elastoplastic version of the Hu–Washizu principle and can be obtained from the simplest elastoplastic Hellinger–Reissner formulation.

We remark that the complementary elastic energy does not appear in the elastoplastic Hellinger–Reissner potential Σ_2 while it is present in the elastic Hellinger–Reissner formulation, see the weak form (33). Accordingly no further difficulty arises in an elastoplastic finite element method based on the two-field Hellinger–Reissener principle compared with the elastoplastic finite element method based on the B-bar three-field variational formulation.

7.1. A numerical simulation

An elastoplastic analysis of the perforated strip shown in Fig. 8 with displacement controlled stretching is investigated by adopting the B-bar formulation and the displacement/pressure approach. A linear elastic behaviour and a linear mixed hardening are considered and the same interpolating spaces for the pressure field and for the elastic volumetric field are assumed so that the limitation principle 7.3 is fulfilled.

The material properties, corresponding to a commercial steel, are Young's modulus $E = 2.1 \times 10^8$ N/m^2, Poisson's ratio $\nu = 0.3$, initial yield threshold $\sigma_0 = 2.4 \times 10^8$ N/m^2. The external load is represented by an imposed displacement of 20 cm. Only a quarter of strip has been considered for symmetry reasons.

The equivalent plastic strains are evaluated at the Gauss points of the element 1 of Fig. 8. In Fig. 9 the equivalent plastic strains are reported in the case of perfect plasticity ($H_{\text{kin}} = H_{\text{iso}} = 0$ N/m^2) and in Fig. 10 the case of mixed hardening with hardening moduli $H_{\text{kin}} = H_{\text{iso}} = 10^7$ N/m^2 is addressed. In agreement with the limitation principle 7.3, the results of the numerical analysis obtained from the two mixed formulations turn out to be coincident.

8. Closure

A general treatment of the variational formulations associated with the generalized elastic material is provided and the complete set of limitation principles for the generalized elastic material is presented. This approach allows us to specialize these principles to elastic and elastoplastic models without repeating ad hoc reasoning.

In the B-bar method the same interpolating spaces are usually assumed in the literature for the pressure field and for the volumetric strain field. Actually this choice fulfils a limitation principle reported in the paper so that there is no additional accuracy in terms of displacement and pressure fields if the (three-field) B-bar method is used instead of the displacement/pressure formulation.

Moreover an analytical condition is proved which ensures the coincidence, in terms of displacements, of the approximate solutions obtained from the mixed displacement/pressure approach and from the displacement method.

The specialization of a limitation principle to an elastoplastic behaviour (GSM) with mixed

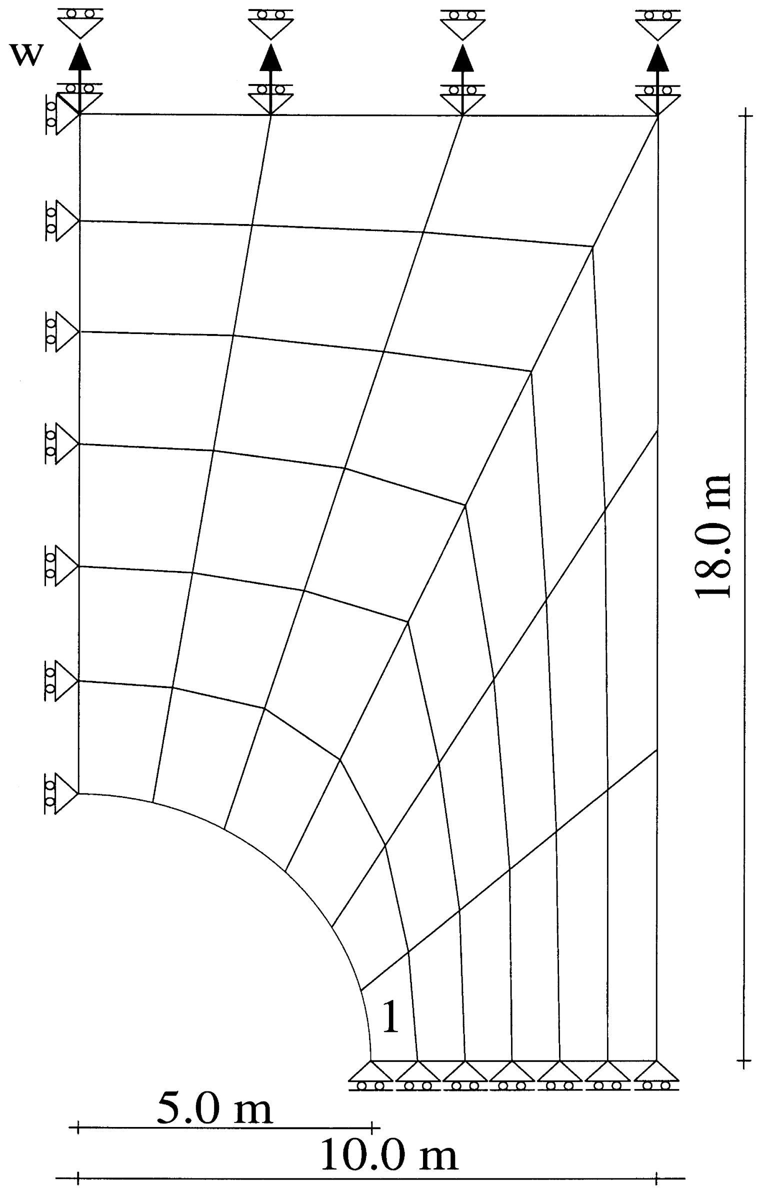

Fig. 8. A quarter of an elastoplastic perforated strip with displacement controlled stretching.

hardening is provided starting from the corresponding principle pertaining to the GEM. It is further shown that some interpolating fields usually adopted in the computational literature for the elastoplastic B-bar mixed finite element discretizations fulfil a limitation principle. Accordingly a two-field mixed formulation can be successfully adopted instead of a three-field approach.

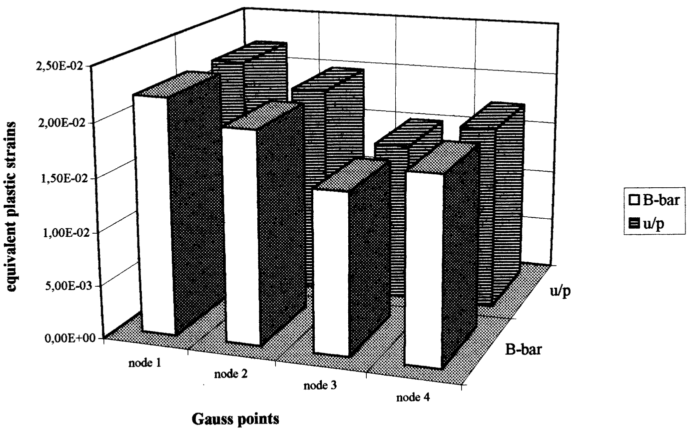

Fig. 9. The equivalent plastic strains at the Gauss points of the element *n*. 1 of Fig. 8 evaluated by adopting the B-bar method and the u/p formulation under the validity of the limitation principle. Elastic-perfectly plastic behaviour.

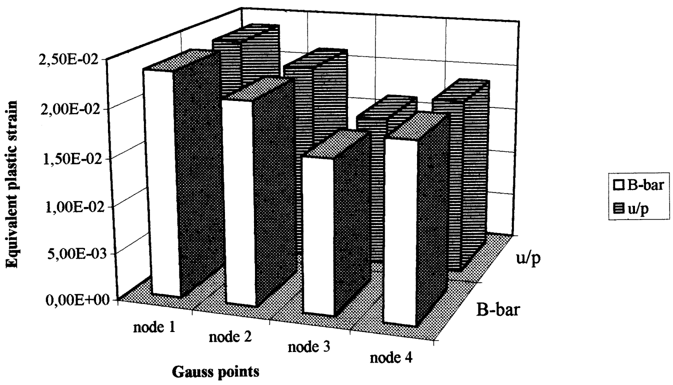

Fig. 10. The equivalent plastic strains at the Gauss points of the element *n*. 1 of Fig. 8 evaluated by adopting the B-bar method and the u/p formulation under the validity of the limitation principle. Elastoplastic behaviour with mixed hardening.

Great care must then be paid to the choice of the interpolating spaces in mixed formulations and limitation principles provide a guideline to check if no additional accuracy is obtained by adopting a mixed formulation with a greater number of unknown fields.

Numerical examples are reported to provide the effectiveness of the limitation principles in elasticity and in elastoplasticity with mixed hardening.

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Appendix I—Some results of convex analysis

Let us recall some basic definitions and properties of convex analysis, e.g., Moreau (1966), Rockafellar (1970), Hiriart-Urruty and Lemaréchal (1993) which are used in the paper.

Let (X, X') be a pair of Hilbert spaces placed in separating duality by a bilinear form $\langle \cdot , \cdot \rangle$ and consider the convex functional $g: X \mapsto \Re \cup \{+\infty\}$. We shall denote by $\bar{\Re}$ the set $\{-\infty\} \cup \Re \cup \{+\infty\}$.

• A graph $\mathscr{G}(M)$ is said to be monotone if:

$$\langle x_2^* - x_1^*, x_2 - x_1 \rangle \geqslant 0 \quad \forall (x_1^*, x_1), (x_2^*, x_2) \in \mathscr{G}(M).$$

The graph $\mathscr{G}(M)$ is maximal monotone, if no point $\{x, x^*\}$ can be added to the graph without violating the monotonicity property.

• A sublinear functional $g: X \mapsto \bar{\Re}$ meets the following properties:

$$\begin{aligned} &g(\alpha x) = \alpha g(x) && \forall \alpha \geqslant 0 && \text{(positive homogeneity)} \\ &g(x_1) + g(x_2) \geqslant g(x_1 + x_2) && \forall x_1, x_2 \in X && \text{(subadditivity)} \end{aligned}$$

• The one-sided Gateaux derivative of g at the point x_0 belonging to the domain of g along the direction given by the vector $x \in X$, is the sublinear functional $f: X \mapsto \bar{\Re}$ defined by:

$$f(x) = \mathrm{d}g(x_0; x) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} [g(x_0 + \varepsilon_x) - g(x_0)].$$

• If the sublinear functional f is proper, the subdifferential of the functional g is the multi-valued map ∂g: $X \mapsto X'$, defined by:

$$\partial g(x_0) = \{x^* \in X' : f(x) \geqslant \langle x^*, x \rangle \quad \forall x \in X\}.$$

In particular, if the functional g is differentiable at $x_0 \in X$, the subdifferential is a singleton and coincides with the usual differential.

• The conjugate of a convex functional g is the convex functional g^*: $X' \mapsto \Re \bigcup \{+\infty\}$ defined by:

$$g^*(x^*) = \sup_{y \in X} \{\langle x^*, y \rangle - g(y)\}.$$

The pairs (x, x^*) for which the 'sup' is attained are said to be conjugate and, provided that g is closed, i.e. $g^{**} = g$, the following Fenchel's relations are equivalent:

$$x \in \partial g^*(x^*) \qquad x^* \in \partial g(x) \qquad g(x) + g^*(x^*) = \langle x^*, x \rangle.$$

• A functional $g: X \mapsto \bar{\Re}$ is lower semicontinuous if:

$$\liminf_{\bar{x} \to x} g(\bar{x}) \geqslant g(x) \quad \forall \bar{x} \in X.$$

A convex functional g is closed, i.e. $g^{**} = g$, if and only if it is l.s.c.

A relevant example of conjugate functionals is provided by the indicator $\sqcup_K$ and the support functional $\sqcup_K^*$ of a convex set K:

$$\sqcup_K(x) = \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{otherwise} \end{cases} \qquad \sqcup_K^*(x^*) = \sup_{x \in K} \langle x^*, x \rangle.$$

• Given two convex functionals g_1: $X \mapsto \Re \bigcup \{+\infty\}$ and g_2: $X \mapsto \Re \bigcup \{+\infty\}$ which are sub-differentiable at $x \in X$, it turns out to be:

$$\partial(g_1 + g_2)(x) = \partial g_1(x) + \partial g_2(x).$$

Analogous results hold for concave functionals by interchanging the role of $+\infty$, $\geqslant$, 'sub' and 'sup' with those of $-\infty$, $\leqslant$, 'super' and 'inf'.

Appendix II—Stationarity conditions

The stationarity of the elastoplastic Hu–Washizu principle reported in Proposition 7.1 provides the following weak equations:

$$\begin{aligned}
&\langle \mathrm{d}g_D^{\mathrm{el}}[P_D(Tu - p)], P_D T\bar{u}\rangle + \langle P_S\sigma, P_S T\bar{u}\rangle = \langle l, \bar{u}\rangle && \forall \bar{u} \in L_0 \\
&\langle P_S\bar{\sigma}, P_S(Tu - e - p_0)\rangle = 0 && \forall P_S\bar{\sigma} \in \mathscr{S} \\
&\langle P_S\bar{\chi}_1, P_S(\alpha_1 - \alpha_{1_0})\rangle = 0 && \forall P_S\bar{\chi}_1 \in \mathscr{X}' \\
&\langle \mathrm{d}g_S^{\mathrm{el}}(P_S e), P_S\bar{e}\rangle - \langle P_S\sigma, P_S\bar{e}\rangle = 0 && \forall P_S\bar{e} \in \mathscr{D} \\
&\langle \mathrm{d}g_S^{\mathrm{h}}(P_S\alpha_1), P_S\bar{\alpha}_1\rangle - \langle P_S\chi_1, P_S\bar{\alpha}_1\rangle = 0 && \forall P_S\bar{\alpha}_1 \in \mathscr{X} \\
&\mathrm{d}D_D[\underset{\sim}{P}_D(\underset{\sim}{p} - \underset{\sim}{p}_0); \underset{\sim}{P}_D\bar{\underset{\sim}{p}}] \geqslant \prec \begin{bmatrix} \mathrm{d}g_D^{\mathrm{el}}[P_D(Tu - p)] \\ \mathrm{d}_{P_D\alpha_1} g_D^{\mathrm{h}}(P_D\alpha_1, \alpha_2) \\ \mathrm{d}_{\alpha_2} g_D^{\mathrm{h}}(P_D\alpha_1, \alpha_2) \end{bmatrix}, \begin{bmatrix} P_D\bar{p} \\ -P_D\bar{\alpha}_1 \\ -\alpha_2 \end{bmatrix} \succ && \forall \underset{\sim}{P}_D\bar{\underset{\sim}{p}} \in \mathscr{Q}.
\end{aligned} \tag{46}$$

Relation $(46)_1$ is the weak form of the equilibrium equation in which the deviatoric part of the constitutive elastic relation is enforced in terms of the deviatoric compatibility condition. Relations $(46)_{2-3}$ show that the increments of the plastic strain field $(Tu - e) - p_0$ and of the kinematic internal variable field $\alpha_1 - \alpha_{1_0}$ are deviatoric. Relation $(46)_4$ provides the expression of the spherical part of the constitutive elastic relation. The spherical part of the static internal variable field χ_1 is obtained from $(46)_5$ as the derivative of the hardening potential with respect to the spherical part of the internal variable field α_1. According to relation $(46)_3$, α_1 is deviatoric so that the spherical part of χ_1 must vanish. Relation $(46)_6$ yields the finite-step flow rule between the deviatoric parts of σ, χ_1 and the internal variable χ_2, defined in terms of the constitutive relations, and the increments $P_D(p - p_0)$, $P_D(\alpha_1 - \alpha_{1_0})$ and $\alpha_2 - \alpha_{2_0}$.

The stationarity of the elastoplastic Hellinger–Reissner principle reported in Proposition 7.2 provides the following weak equations:

$$
\begin{aligned}
&\langle \mathrm{d}g_S^{\mathrm{el}}[P_S(Tu-p)], P_S T\bar{u}\rangle + \langle \mathrm{d}g_D^{\mathrm{el}}[P_D(Tu-p)], P_D T\bar{u}\rangle = \langle l, \bar{u}\rangle && \forall \bar{u} \in L_0 \\
&\langle P_S\bar{\sigma}, P_S(p-p_0)\rangle = 0 && \forall P_S\bar{\sigma} \in \mathscr{S} \\
&\langle P_S\bar{\chi}_1, P_S(\alpha_1-\alpha_{1_0})\rangle = 0 && \forall P_S\bar{\chi}_1 \in \mathscr{X}' \\
&\langle \mathrm{d}g_S^{\mathrm{el}}[P_S(Tu-p)], P_S\bar{p}\rangle - \langle P_S\sigma, P_S\bar{p}\rangle = 0 && \forall P_S\bar{p} \in \mathscr{D} \\
&\langle \mathrm{d}g_S^{\mathrm{h}}(P_S\alpha_1), P_S\bar{\alpha}_1\rangle - \langle P_S\chi_1, P_S\bar{\alpha}_1\rangle = 0 && \forall P_S\bar{\alpha}_1 \in \mathscr{X} \\
&\mathrm{d}D_D[\underset{\sim}{P}_D(\underset{\sim}{p}-\underset{\sim}{p}_0); \underset{\sim}{P}_D\bar{\underset{\sim}{p}}] \geqslant \prec \begin{bmatrix} \mathrm{d}g_D^{\mathrm{el}}[P_D(Tu-p)] \\ \mathrm{d}_{P_D\alpha_1} g_D^{\mathrm{h}}(P_D\alpha_1,\alpha_2) \\ \mathrm{d}_{\alpha_2} g_D^{\mathrm{h}}(P_D\alpha_1,\alpha_2) \end{bmatrix}, \begin{bmatrix} P_D\bar{p} \\ -P_D\bar{\alpha}_1 \\ -\alpha_2 \end{bmatrix} \succ && \forall \underset{\sim}{P}_D\bar{\underset{\sim}{p}} \in \underset{\sim}{\mathscr{D}}.
\end{aligned}
\tag{47}
$$

Relation $(47)_1$ is the weak form of the equilibrium equation where the spherical and deviatoric constitutive elastic relations in terms of compatible elastic strain fields are enforced. Relations $(47)_{2-3}$ show that the increments of the plastic strain field $p-p_0$ and of the kinematic internal variable field $\alpha_1-\alpha_{1_0}$ are deviatoric. Relation $(47)_4$ provides the expression of the spherical part of the stress field σ. The spherical part of the static internal variable field χ_1 is obtained from $(47)_5$ as the derivative of the hardening potential with respect to the spherical part of the internal variable field α_1. According to relation $(47)_3$, the spherical part of χ_1 must vanish being α_1 deviatoric. Relation $(47)_6$ yields the finite-step flow rule between the deviatoric parts of σ, χ_1 and the expression of χ_2, defined in terms of the constitutive relations, and the increments $P_D(p-p_0)$, $P_D(\alpha_1-\alpha_{1_0})$ and $\alpha_2-\alpha_{2_0}$.

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